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The Weyl groups and weight multiplicities of the exceptional Lie groups

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Abstract. The Weyl group, W_G, of each exceptional simple Lie group G, is described in detail. Its structure is defined in terms of its coset decomposition with respect to the Weyl group, W_H, of a classical semi-simple Lie group, H, embedded naturally in G. The concepts of G-dominance and G-equivalence are defined and used to determine, from the character formula of Weyl, the branching rule associated with the restriction of group elements from G to H. The Weyl group W_G is used further to impose constraints on both the branching multiplicities for $G \to H$ and the weight multiplicities of G. These constraints are used to evaluate the weight multiplicities of F_4, E_6, E_7 and E_8 together with the branching multiplicities for $E_8 \to SO(16)$.

1. Introduction

As stressed by Wybourne and Bowick (1977) a number of recent applications of the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 to physics have made it necessary to establish results on the irreducible representations of these groups analogous to those appropriate to the classical Lie groups $SU(k+1)$, $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$. Amongst other requirements for these applications are the evaluation of weight and branching multiplicities. In this paper it is shown that these two types of multiplicity are intimately related and that the Weyl symmetry groups of the exceptional Lie groups have a special role to play in their evaluation.

It is particularly important to make use of the relation between the Weyl symmetry groups of each exceptional Lie group and a naturally embedded classical Lie subgroup. Such natural embeddings have been discussed in detail in the preceding paper (King and Al-Qubanchi 1981), hereafter referred to as I. It has been shown in I that each natural embedding serves to define a natural labelling scheme for the irreducible representations of the exceptional Lie groups. In this paper the labelling schemes based on the embeddings $G_2 \supset SU(3)$, $F_4 \supset SO(9)$, $E_6 \supset SU(2) \otimes SU(6)$, $E_7 \supset SU(8)$ and $E_8 \supset SO(16)$ are used. The advantages of these particular choices are explained in the following section in which the Weyl group W_G of each exceptional Lie group G is completely determined from the results of I and expressed in terms of its coset structure with respect to the Weyl group W_H of the appropriate classical Lie subgroup H of G.

In § 3 the concepts of G-dominance and G-equivalence of vectors are defined in a notation appropriate to both the specification of weights of irreducible representations and the labelling of those irreducible representations. These concepts are used in § 4 in conjunction with the character formula due to Weyl (1926) to determine algorithms for

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the determination of the branching multiplicities associated with the restriction G → H. These algorithms depend for their implementation solely upon the properties of the classical Lie group H. Unfortunately they are rather inefficient. The situation is remedied in § 5 in which it is shown that the G-equivalence of weights of an exceptional Lie group G imposes strong constraints on both the weight multiplicities of G and simultaneously the branching multiplicities for G → H. These constraints are such that both such multiplicities may be calculated in terms of the known weight multiplicities of the classical Lie group H (King and Plunkett 1976). In this way tabulations of the weight multiplicities of F_4, E_6, E_7 and E_8 are built up. The work of Wybourne and Bowick (1977) precludes the necessity of tabulating the branching multiplicities other than those for the restriction $E_8 \rightarrow SO(16)$. These results complete the task initiated earlier for the group G_2 (King and Al-Qubanchi 1978).

2. The Weyl group

The system of roots, Σ_g, of a complex semi-simple Lie algebra g, of rank k, is a set of vectors $\boldsymbol{r}$ in k-dimensional root space V. This space, as explained in I, may conveniently be embedded in a Euclidean space W of dimension d with $d \geqslant k$. The orthogonal complement, $V^{\perp}$, of V in W is spanned by certain vectors $\boldsymbol{p}$ belonging to a complementary set Γ_g. These vectors satisfy the constraint $\boldsymbol{r} \cdot \boldsymbol{p} = 0$ for all root vectors $\boldsymbol{r}$. The basis vectors of W are the mutually orthonormal vectors $\boldsymbol{e}_i$ with $i = 1, 2, \ldots, d$. They are defined so that the jth component of $\boldsymbol{e}_i$ is given by $(\boldsymbol{e}_i)_j = \delta_{ij}$. An ordering of vectors in W may be introduced such that $\boldsymbol{v}$ is higher than $\boldsymbol{w}$, signified by $\boldsymbol{v} > \boldsymbol{w}$, if and only if the first non-vanishing component of $\boldsymbol{v} - \boldsymbol{w}$ with respect to the basis defined by $\boldsymbol{e}_i$, $i = 1, 2, \ldots, d$, is positive.

A root $\boldsymbol{r}$ is positive if $\boldsymbol{r} > \boldsymbol{0}$, where all the components of $\boldsymbol{0}$ are zero. Furthermore, a root is said to be simple if it is positive and may not be written as the sum of two positive roots. Such simple roots were introduced by Dynkin (1962, p 432), who proved that each complex semi-simple Lie algebra g is characterised by its system, Π_g, of simple roots.

As pointed out in I, there is a consensus of opinion regarding the specification of the roots, Σ_g, and the simple roots, Π_g, of each of the simple classical Lie algebras of rank k: A_k, B_k, C_k and D_k associated with the classical groups $SU(k+1)$, $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ respectively. No such consensus has emerged in the case of the simple exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 associated with the exceptional groups denoted by the same symbols. This is primarily because the natural way of constructing the root system of an exceptional simple Lie algebra g, of rank k, depends upon an embedding in g of a classical semi-simple Lie subalgebra h of the same rank k. For each g there may exist more than one such h, leading therefore to the great variety of root systems for g appearing in the literature. This has been discussed in detail in I. For reasons which will become clear the labelling schemes adhered to in this paper are those based on the embeddings: $G_2 \supset A_2$, $F_4 \supset B_4$, $E_6 \supset A_1 \dot{+} A_5$, $E_7 \supset A_7$ and $E_8 \supset D_8$. The corresponding root systems, along with those of the classical simple algebras, are displayed in table 1. For each algebra the important quantity

$$\boldsymbol{R} = \tfrac{1}{2} \sum_{\boldsymbol{r} > \boldsymbol{0}} \boldsymbol{r} \tag{2.1}$$

is also given, as well as the vectors $\boldsymbol{p}$ which define $V^{\perp}$.

Table 1.

Lie group G	Lie algebra g	Complementary system $\boldsymbol{p} \in \Gamma_g$	Root system $\boldsymbol{r} \in \Sigma_g$		$\boldsymbol{R} = \frac{1}{2} \sum_{r>0} \boldsymbol{r}$
SU($k+1$)	A_k	$e_1 + e_2 + \ldots + e_{k+1}$	$\pm(e_i - e_j)$	$1 \leqslant i < j \leqslant k+1$	$\frac{1}{2}(k, k-2, \ldots, -k+2, -k)$
SO($2k+1$)	B_k	—	$\pm e_i$ $\pm e_i \pm e_j$	$1 \leqslant i \leqslant k$ $1 \leqslant i < j \leqslant k$	$\frac{1}{2}(2k-1, 2k-3, \ldots, 3, 1)$
Sp($2k$)	C_k	—	$\pm 2e_i$ $\pm e_i \pm e_j$	$1 \leqslant i \leqslant k$ $1 \leqslant i < j \leqslant k$	$(k, k-1, \ldots, 2, 1)$
SO($2k$)	D_k	—	$\pm e_i \pm e_j$	$1 \leqslant i < j \leqslant k$	$(k-1, k-2, \ldots, 1, 0)$
G_2	G_2	$e_1 + e_2 + e_3$	$\pm(e_i - e_j)$ $\pm\frac{1}{3}(2e_i - e_j - e_k)$	$1 \leqslant i < j \leqslant 3$ $1 \leqslant i \leqslant 3, \quad 1 \leqslant j < k \leqslant 3, \quad i \neq j, k$	$\frac{1}{3}(5, -1, -4)$
F_4	F_4	—	$\pm e_i$ $\pm e_i \pm e_j$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$	$1 \leqslant i \leqslant 4$ $1 \leqslant i < j \leqslant 4$	$\frac{1}{2}(11, 5, 3, 1)$
E_6	E_6	$e_1 + e_8$ $e_2 + e_3 + \ldots + e_7$	$\pm(e_1 - e_8)$ $\pm(e_i - e_j)$ $\frac{1}{2}[\pm(e_1 - e_8) \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7]$	 $2 \leqslant i < j \leqslant 7$ (4 + signs, 4 − signs)	$\frac{1}{2}(11, 5, 3, 1, -1, -3, -5, -11)$
E_7	E_7	$e_1 + e_2 + \ldots + e_8$	$\pm(e_i - e_j)$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$	$1 \leqslant i < j \leqslant 8$ (4 + signs, 4 − signs)	$\frac{1}{4}(49, 5, 1, -3, -7, -11, -15, -19)$
E_8	E_8	—	$\pm e_i \pm e_j$ $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$	$1 \leqslant i < j \leqslant 8$ (even no. of +signs)	$(23, 6, 5, 4, 3, 2, 1, 0)$

The Weyl group, W_G, of a real compact semi-simple Lie group G is the symmetry group of the root diagram of the associated complex semi-simple Lie algebra g. It is the group generated by reflections in the hyperplanes perpendicular to the roots. The action of such a reflection S_r on an arbitrary vector w in the Euclidean space W, containing the root space, is defined by:

$$S_r: \boldsymbol{w} \to S_r\boldsymbol{w} = \boldsymbol{w} - 2[(\boldsymbol{w} \cdot \boldsymbol{r})/(\boldsymbol{r} \cdot \boldsymbol{r})]\boldsymbol{r}. \tag{2.2}$$

Clearly $S_{-r}\boldsymbol{w} = S_r\boldsymbol{w}$ and more generally $S_{ar}\boldsymbol{w} = S_r\boldsymbol{w}$ for any $a \neq 0$. These reflections preserve both $V^\perp$ and V in the sense that for all $\boldsymbol{p} \in V^\perp$, $S_r\boldsymbol{p} = \boldsymbol{p}$ by virtue of the constraint $\boldsymbol{p} \cdot \boldsymbol{r} = 0$, whilst if $\boldsymbol{v} \in V$ so that $\boldsymbol{v} \cdot \boldsymbol{p} = 0$ for all $\boldsymbol{p} \in V^\perp$ then $S_r\boldsymbol{v} \in V$ since $(S_r\boldsymbol{v}) \cdot \boldsymbol{p} = 0$ for all $\boldsymbol{p} \in V^\perp$.

In the case of the roots of the classical Lie algebras:

$$S_{e_i-e_j}: \boldsymbol{w} \to \boldsymbol{w} - (w_i - w_j)(\boldsymbol{e}_i - \boldsymbol{e}_j)$$

$$S_{e_i+e_j}: \boldsymbol{w} \to \boldsymbol{w} - (w_i + w_j)(\boldsymbol{e}_i + \boldsymbol{e}_j)$$

$$S_{e_i}: \boldsymbol{w} \to \boldsymbol{w} - 2w_i\boldsymbol{e}_i,$$

so that for $\boldsymbol{v} = (v_1, v_2, \ldots, v_d) \in V$:

$$S_{e_i-e_j}: (\ldots v_i \ldots v_j \ldots) \to (\ldots v_j \ldots v_i \ldots) \tag{2.3}$$

$$S_{e_i+e_j}: (\ldots v_i \ldots v_j \ldots) \to (\ldots -v_j \ldots -v_i \ldots) \tag{2.4}$$

$$S_{e_i}: (\ldots v_i \ldots) \to (\ldots -v_i \ldots) \tag{2.5}$$

where the components of the vector $\boldsymbol{v}$ indicated by dots are unchanged by the reflection.

It is then easy to identify, for each of the classical simple Lie groups, the corresponding Weyl groups formed by closure under successive application of the relevant transformations selected from (2.3), (2.4) and (2.5). In the case of SU$(k+1)$ only (2.3) is relevant and $W_{SU(k+1)}$ is the group, of order $(k+1)!$, of all permutations of the components of $\boldsymbol{v}$. For SO$(2k+1)$ and Sp$(2k)$ (2.3), (2.4) and (2.5) are all relevant. The Weyl groups $W_{SO(2k+1)}$ and $W_{Sp(2k)}$ are identical, each being the group, of order $2^k k!$, of all permutations and independent sign changes of the components of $\boldsymbol{v}$. On the other hand for SO$(2k)$, (2.5) is not relevant and $W_{SO(2k)}$ is the group, of order $2^{k-1}k!$, consisting of all permutations and an even number of independent sign changes of the components of $\boldsymbol{v}$.

For each exceptional simple Lie group G one advantage of using a natural labelling scheme for the roots of the corresponding exceptional simple Lie algebra g, as described in I, now becomes apparent. Such a scheme makes it clear that the root system, Σ_g, of g contains the root system, Σ_h, of a classical semi-simple Lie algebra h, where h corresponds to a classical subgroup H of G. It follows that W_G contains W_H as a subgroup, and it is convenient to construct W_G by making use of its coset decomposition with respect to W_H, namely:

$$W_G = \bigcup_{\gamma=1}^{c} W_H S_\gamma \tag{2.6}$$

where $c = |W_G|/|W_H|$ and S_γ for $\gamma = 1, 2, \ldots, c$ are a set of coset representatives such that each element S of W_G can be written in the form TS_γ for some element T in W_H and some coset representative S_γ. It is at this stage that the criterion used in selecting from the set of all natural labelling schemes the particular ones of table 1 becomes apparent. This criterion is that of the minimisation of the index c. The embeddings $G_2 \supset$ SU(3),

$F_4 \supset SO(9)$, $E_6 \supset SU(2) \otimes SU(6)$, $E_7 \supset SU(8)$ and $E_8 \supset SO(16)$ correspond to the minimal possible values of c in (2.6), namely $c = 2, 3, 36, 72$ and 135 respectively.

To determine suitable coset representatives S_γ it is necessary to consider the Weyl reflections associated with those roots of g which are not roots of h. In the case of G_2, for example, the roots additional to those of A_2 define reflections of the form:

$$S_{(2e_i - e_j - e_k)/3} : \boldsymbol{w} \to \boldsymbol{w} - (2w_i - w_j - w_k)\tfrac{1}{3}(2\boldsymbol{e}_i - \boldsymbol{e}_j - \boldsymbol{e}_k).$$

The space $V^\perp$ is spanned by $\boldsymbol{p} = \boldsymbol{e}_i + \boldsymbol{e}_2 + \boldsymbol{e}_3$, so that for $\boldsymbol{v} = (v_1, v_2, v_3)$ in V, $\boldsymbol{v} \cdot \boldsymbol{p} = v_1 + v_2 + v_3 = 0$. The use of this constraint gives:

$$S_{(2e_i - e_j - e_k)/3} : (v_i, v_j, v_k) \to (-v_i, -v_k, -v_j). \tag{2.7}$$

Making use of the known structure of the subgroup $W_{SU(3)}$ it follows that W_{G_2} is the group of order 12, consisting of all permutations and simultaneous sign changes of all three components of $\boldsymbol{v}$. Thus the index of $W_{SU(3)}$ in W_{G_2} is 2, as claimed, and

$$W_{G_2} = W_{SU(3)} S_1 \cup W_{SU(3)} S_2$$

where the coset representatives S_1 and S_2 may be chosen to be the identity element:

$$S_1 : (v_1, v_2, v_3) \to (v_1, v_2, v_3)$$

and the reflection:

$$S_2 = S_{(e_1 - 2e_2 + e_3)/3} : (v_1, v_2, v_3) \to (-v_3, -v_2, -v_1). \tag{2.8}$$

Similarly, in the case of F_4, E_6, E_7 and E_8 the roots additional to those of the classical subalgebras B_4, $A_1 \dotplus A_5$, A_7 and D_8, respectively, define reflections of the form:

$$S_{\sum_j \sigma_j e_j/2} : \boldsymbol{w} \to \boldsymbol{w} - \Big(\sum_i \sigma_i w_i\Big)\tfrac{1}{2} \sum_j \sigma_j \boldsymbol{e}_j$$

with $\sigma_j = \mp 1$ for $j = 1, 2, \ldots, d$. Thus for any vector $\boldsymbol{v}$ in V

$$S_{\sum_j \sigma_j e_j/2} : (\ldots v_i \ldots) \to \Big(\ldots \tfrac{1}{2}(v_i - \sigma_i \sum_{j \neq i} \sigma_j v_j) \ldots\Big) \tag{2.9}$$

where now the components of the vector $\boldsymbol{v}$ indicated by dots transform in the same way as the particular component v_i for which the transformation has been given explicitly. The reflection (2.9) may be simplified where appropriate by use of the constraints $\boldsymbol{v} \cdot \boldsymbol{p} = 0$. In order to construct explicitly the corresponding Weyl groups it is necessary to establish the result of repeated application of (2.9) together with (2.3), (2.4) and (2.5) as appropriate until closure is obtained. This has been carried out explicitly elsewhere (Al-Qubanchi 1978) and confirms that the Weyl groups of F_4, E_6, E_7 and E_8 are groups of order $3(2^4 4!)$, $36(2!6!)$, $72(8!)$ and $135(2^7 8!)$. Convenient sets of coset representatives appropriate to subgroups consisting of the Weyl groups of $SO(9)$, $SU(2) \otimes SU(6)$, $SU(7)$ and $SO(16)$ are given in table 2, in which the Weyl group W_G of each simple Lie group G is displayed.

This table provides not only an explicit statement of the action of each Weyl group element S on an arbitrary vector $\boldsymbol{v}$ in the root space V, but also the parity, η_S, of this element. The parity η_S is $+1$ or -1 according to whether S is generated by an even or an odd number of reflections of the type (2.2).

Table 2.

Lie group G	Order of Weyl group $\lvert W_G \rvert$	Restriction on components of $v \in V$	Action of $S \in W_G$ on $v \in V$ Components of Sv	Components of $S_\gamma v$	Parity η_S
$SU(k+1)$	$(k+1)!$	$v_1+v_2+\ldots+v_{k+1}=0$	$(v_{\pi 1}, v_{\pi 2}, \ldots, v_{\pi k+1})$ $\pi = \begin{pmatrix} 1 & 2 & \ldots & k+1 \\ \pi_1 & \pi_2 & \ldots & \pi_{k+1} \end{pmatrix}$		$(-)^\pi$
$SO(2k+1)$	$2^k . k!$	—	$(\sigma_1 v_{\pi 1}, \sigma_2 v_{\pi 2}, \ldots, \sigma_k v_{\pi k})$ $\pi = \begin{pmatrix} 1 & 2 \ldots k \\ \pi_1 \pi_2 \ldots \pi_k \end{pmatrix}$ $\sigma_i = \pm 1$ for $i = 1, 2, \ldots, k$ $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k = \pm 1$		$\sigma(-)^\pi$
$Sp(2k)$	$2^k . k!$	—	$(\sigma_1 v_{\pi 1}, \sigma_2 v_{\pi 2}, \ldots, \sigma_k v_{\pi k})$ $\pi = \begin{pmatrix} 1 & 2 \ldots k \\ \pi_1 \pi_2 \ldots \pi_k \end{pmatrix}$ $\sigma_i = \pm 1$ for $i = 1, 2, \ldots, k$ $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k = \pm 1$		$\sigma(-)^\pi$
$SO(2k)$	$2^{k-1} . k!$	—	$(\sigma_1 v_{\pi 1}, \sigma_2 v_{\pi 2}, \ldots, \sigma_k v_{\pi k})$ $\pi = \begin{pmatrix} 1 & 2 \ldots k \\ \pi_1 \pi_2 \ldots \pi_k \end{pmatrix}$ $\sigma_i = \pm 1$ for $i = 1, 2, \ldots, k$ $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k = 1$		$(-)^\pi$
G_2	$2 . 3!$	$v_1+v_2+v_3=0$	$Sv = TS_\gamma v$ with $T \in W_{SU(3)}$ (i) $\gamma = 1$ (ii) $\gamma = 2$	(i) (ii) v_1 $-v_3$ v_2 $-v_2$ v_3 $-v_1$	(i) $(-)^\pi$ (ii) $-(-)^\pi$
F_4	$3 . 2^4 . 4!$	—	$Sv = TS_\gamma v$ with $T \in W_{SO(9)}$ (i) $\gamma = 1$ (ii) $\gamma = 2, 3$ $\tau = \pm 1$	(i) (ii) v_1 $\frac{1}{2}(v_1+v_2+v_3+\tau v_4)$ v_2 $\frac{1}{2}(v_1+v_2-v_3-\tau v_4)$ v_3 $\frac{1}{2}(v_1-v_2+v_3-\tau v_4)$ v_4 $\frac{1}{2}(v_1-v_2-v_3+\tau v_4)$	(i) $\sigma(-)^\pi$ (ii) $-\sigma\tau(-)^\pi$

E_6	$36.2!6!$	$v_1+v_8=0$	$\boldsymbol{Sv=TS_\gamma v}$ with	(i)	(ii)	(iii)		(i) $(-)^\pi$
			$T\in W_{SU(2)}\times W_{SU(6)}$	v_1	$\frac{1}{2}(v_1-v_{\rho5}-v_{\rho6}-v_{\rho7})$	$\frac{1}{2}(v_{\rho2}-v_{\rho7})$		(ii) $-(-)^\pi(-)^\rho$
		$v_2+v_3+\ldots+v_7=0$	(i) $\gamma=1$	v_2	$\frac{1}{2}(v_1+v_{\rho2}-v_{\rho3}-v_{\rho4})$	$v_1+\frac{1}{2}(v_{\rho2}+v_{\rho7})$		(iii) $(-)^\pi(-)^\rho$
			(ii) $\gamma=2,3,\ldots,21$	v_3	$\frac{1}{2}(v_1-v_{\rho2}+v_{\rho3}-v_{\rho4})$	$\frac{1}{2}(v_{\rho3}+v_{\rho4}+v_{\rho5}-v_{\rho6})$		
			$\rho=\begin{pmatrix}2 & 3 & \ldots & 7\\ \rho_2 & \rho_3 & \ldots & \rho_7\end{pmatrix}$	v_4	$\frac{1}{2}(v_1-v_{\rho2}-v_{\rho3}+v_{\rho4})$	$\frac{1}{2}(v_{\rho3}+v_{\rho4}-v_{\rho5}+v_{\rho6})$		
			$2\leqslant\rho_2<\rho_3<\rho_4\leqslant7$	v_5	$\frac{1}{2}(-v_1+v_{\rho5}-v_{\rho6}-v_{\rho7})$	$\frac{1}{2}(v_{\rho3}-v_{\rho4}+v_{\rho5}+v_{\rho6})$		
			$2\leqslant\rho_5<\rho_6<\rho_7\leqslant7$	v_6	$\frac{1}{2}(-v_1-v_{\rho5}+v_{\rho6}-v_{\rho7})$	$\frac{1}{2}(-v_{\rho3}+v_{\rho4}+v_{\rho5}+v_{\rho6})$		
			(iii) $\gamma=22,23,\ldots,36$	v_7	$\frac{1}{2}(-v_1-v_{\rho5}-v_{\rho6}+v_{\rho7})$	$-v_1+\frac{1}{2}(v_{\rho2}+v_{\rho7})$		
			$\rho=\begin{pmatrix}2 & 3 & \ldots & 7\\ \rho_2 & \rho_3 & \ldots & \rho_7\end{pmatrix}$	v_8	$\frac{1}{2}(-v_1+v_{\rho5}+v_{\rho6}+v_{\rho7})$	$\frac{1}{2}(-v_{\rho2}+v_{\rho7})$		
			$2\leqslant\rho_2<\rho_7\leqslant7$					
			$2\leqslant\rho_3<\rho_4<\rho_5<\rho_6\leqslant7$					
E_7	$72.8!$	$v_1+v_2+\ldots+v_8=0$	$\boldsymbol{Sv=TS_\gamma v}$ with $T\in W_{SU(8)}$	(i)	(ii)	(iii)	(iv)	
			(i) $\gamma=1$	v_1	$-v_8$	$\frac{1}{2}(v_1-v_{\rho6}-v_{\rho7}-v_{\rho8})$	$\frac{1}{2}(v_1+v_{\rho2}+v_{\rho3}-v_{\rho4})$	(i) $(-)^\pi$
			(ii) $\gamma=2$	v_2	$-v_7$	$\frac{1}{2}(v_{\rho2}-v_{\rho3}-v_{\rho4}-v_{\rho5})$	$\frac{1}{2}(v_1+v_{\rho2}-v_{\rho3}+v_{\rho4})$	(ii) $-(-)^\pi$
			(iii) $\gamma=3,4,\ldots,37$	v_3	$-v_6$	$\frac{1}{2}(-v_{\rho2}+v_{\rho3}-v_{\rho4}-v_{\rho5})$	$\frac{1}{2}(v_1-v_{\rho2}+v_{\rho3}+v_{\rho4})$	(iii) $-(-)^\pi(-)^\rho$
			$\rho=\begin{pmatrix}2 & 3 & \ldots & 8\\ \rho_2 & \rho_3 & \ldots & \rho_8\end{pmatrix}$	v_4	$-v_5$	$\frac{1}{2}(-v_{\rho2}-v_{\rho3}+v_{\rho4}-v_{\rho5})$	$\frac{1}{2}(v_{\rho5}+v_{\rho6}+v_{\rho7}-v_{\rho8})$	(iv) $(-)^\pi(-)^\rho$
			$2\leqslant\rho_2<\rho_3<\rho_4<\rho_5\leqslant8$	v_5	$-v_4$	$\frac{1}{2}(-v_{\rho2}-v_{\rho3}-v_{\rho4}+v_{\rho5})$	$\frac{1}{2}(v_{\rho5}+v_{\rho6}-v_{\rho7}+v_{\rho8})$	
			$2\leqslant\rho_6<\rho_7<\rho_8\leqslant8$	v_6	$-v_3$	$\frac{1}{2}(-v_1+v_{\rho6}-v_{\rho7}-v_{\rho8})$	$\frac{1}{2}(v_{\rho5}-v_{\rho6}+v_{\rho7}+v_{\rho8})$	
			(iv) $\gamma=38,39,\ldots,72$	v_7	$-v_2$	$\frac{1}{2}(-v_1-v_{\rho6}+v_{\rho7}-v_{\rho8})$	$\frac{1}{2}(-v_{\rho5}+v_{\rho6}+v_{\rho7}+v_{\rho8})$	
			$\rho=\begin{pmatrix}2 & 3 & \ldots & 8\\ \rho_2 & \rho_3 & \ldots & \rho_8\end{pmatrix}$	v_8	$-v_1$	$\frac{1}{2}(-v_1-v_{\rho6}-v_{\rho7}+v_{\rho8})$	$\frac{1}{2}(-v_1+v_{\rho2}+v_{\rho3}+v_{\rho4})$	
			$2\leqslant\rho_2<\rho_3<\rho_4\leqslant8$					
			$2\leqslant\rho_5<\rho_6<\rho_7<\rho_8\leqslant8$					

Table 2—continued

Lie group G	Order of Weyl group $\|W_G\|$	Action of $S \in W_G$ on $\boldsymbol{v} \leftarrow V$ Components of $S\boldsymbol{v}$	Components of $S_\gamma \boldsymbol{n}$			Parity η_S
E_8	$135.2^7.8!$	$S\boldsymbol{v} = TS_\gamma \boldsymbol{v}$ with $T \in W_{SO(16)}$ (i) $\gamma = 1$ (ii) $\gamma = 2.\quad a = 1$ $\rho = \begin{pmatrix} 2 & 3 & \dots 8 \\ \rho_2 & \rho_3 & \dots \rho_8 \end{pmatrix}$ $2 \leqslant \rho_2 < \rho_3 < \dots < \rho_8 \leqslant 8$ $\gamma = 3, 4, \dots, 23 \quad a = 2$ $\rho = \begin{pmatrix} 2 & 3 & \dots 8 \\ \rho_2 & \rho_3 & \dots \rho_8 \end{pmatrix}$ $2 \leqslant \rho_2 < \rho_3 \leqslant 8$ $2 \leqslant \rho_4 < \rho_5 < \dots < \rho_8 \leqslant 8$ $\gamma = 24, 25, \dots, 58 \quad a = 3$ $\rho = \begin{pmatrix} 2 & 3 & \dots 8 \\ \rho_2 & \rho_3 & \dots \rho_8 \end{pmatrix}$ $2 \leqslant \rho_2 < \rho_3 < \rho_4 < \rho_5 \leqslant 8$ $2 \leqslant \rho_6 < \rho_7 < \rho_8 \leqslant 8$ $\gamma = 59, 60, \dots, 65 \quad a = 4$ $\rho = \begin{pmatrix} 2 & 3 & \dots 8 \\ \rho_2 & \rho_3 & \dots \rho_8 \end{pmatrix}$ $2 \leqslant \rho_2 < \rho_3 < \dots < \rho_7 \leqslant 8$ $2 \leqslant \rho_8 \leqslant 8$ (iii) $\gamma = 66, 67, \dots, 135$ $\tau = \pm 1$ $\rho = \begin{pmatrix} 2 & 3 & \dots 8 \\ \rho_2 & \rho_3 & \dots \rho_8 \end{pmatrix}$ $2 \leqslant \rho_2 < \rho_3 < \rho_4 \leqslant 8$ $2 \leqslant \rho_5 < \rho_6 < \rho_7 < \rho_8 \leqslant 8$	(i) v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8	(ii) $\frac{1}{4}(3v_1 + v_{\rho 2} + v_{\rho 3} \dots + v_{\rho 2a-1} - v_{\rho 2a} - v_{\rho 2a+1} \dots + v_{\rho 7} - v_{\rho 8})$ $\frac{1}{4}(v_1 + 3v_{\rho 2} - v_{\rho 3} \dots - v_{\rho 2a-1} + v_{\rho 2a} + v_{\rho 2a+1} \dots + v_{\rho 7} + v_{\rho 8})$ $\frac{1}{4}(v_1 - v_{\rho 2} + 3v_{\rho 3} \dots - v_{\rho 2a-1} + v_{\rho 2a} + v_{\rho 2a+1} \dots + v_{\rho 7} + v_{\rho 8})$ $\vdots$ $\frac{1}{4}(v_1 - v_{\rho 2} - v_{\rho 3} \dots + 3v_{\rho 2a-1} + v_{\rho 2a} + v_{\rho 2a+1} \dots + v_{\rho 7} + v_{\rho 8})$ $\frac{1}{4}(v_1 - v_{\rho 2} - v_{\rho 3} \dots - v_{\rho 2a-1} + v_{\rho 2a} + v_{\rho 2a+1} \dots + v_{\rho 7} - 3v_{\rho 8})$ $\frac{1}{4}(v_1 - v_{\rho 2} - v_{\rho 3} \dots - v_{\rho 2a-1} + v_{\rho 2a} + v_{\rho 2a+1} \dots - 3v_{\rho 7} + v_{\rho 8})$ $\vdots$ $\frac{1}{4}(v_1 - v_{\rho 2} - v_{\rho 3} \dots - v_{\rho 2a-1} + v_{\rho 2a} - 3v_{\rho 2a+1} \dots + v_{\rho 7} + v_{\rho 8})$ $\frac{1}{4}(-v_1 + v_{\rho 2} + v_{\rho 3} \dots + v_{\rho 2a-1} + 3v_{\rho 2a} - v_{\rho 2a+1} \dots - v_{\rho 7} - v_{\rho 8})$	(iii) $\frac{1}{2}(v_1 + v_{\rho 2} + v_{\rho 3} + \tau v_{\rho 4})$ $\frac{1}{2}(v_1 + v_{\rho 2} - v_{\rho 3} - \tau v_{\rho 4})$ $\frac{1}{2}(v_1 - v_{\rho 2} + v_{\rho 3} - \tau v_{\rho 4})$ $\frac{1}{2}(v_1 - v_{\rho 2} - v_{\rho 3} + \tau v_{\rho 4})$ $\frac{1}{2}(v_{\rho 5} + v_{\rho 6} + v_{\rho 7} + \tau v_{\rho 8})$ $\frac{1}{2}(v_{\rho 5} + v_{\rho 6} - v_{\rho 7} - \tau v_{\rho 8})$ $\frac{1}{2}(v_{\rho 5} - v_{\rho 6} + v_{\rho 7} - \tau v_{\rho 8})$ $\frac{1}{2}(v_{\rho 5} - v_{\rho 6} - v_{\rho 7} + \tau v_{\rho 8})$	(i) $(-)^\pi$ (ii) $-(-1)^a(-)^\pi(-)^\rho$ (iii) $(-)^\pi(-)^\rho$

3. G-dominance

Given a vector $\boldsymbol{v}$ in the root space V of some semi-simple Lie algebra g associated with a Lie group G there exists a set of vectors $\{S\boldsymbol{v}: S \in W_G\}$ each of which may be said to be G-equivalent to $\boldsymbol{v}$, and a unique G-dominant vector $\boldsymbol{v}_G$ which is defined to be the highest vector of this set. Using the information in table 2 it is a straightforward task to construct, from a given vector $\boldsymbol{v}$, the corresponding class of G-equivalent vectors and to identify the G-dominant vector $\boldsymbol{v}_G$.

For example the vector $\boldsymbol{v} = (0, 1, -3, 2)$ in the root space V of A_3 is SU(4)-equivalent to others such as $(1, 0, -3, 2)$, $(-3, 2, 1, 0)$, formed by permuting the components of $\boldsymbol{v}$. The corresponding SU(4)-dominant vector, $\boldsymbol{v}_{SU(4)}$, is clearly $(2, 1, 0, -3)$. Similarly the vector $\boldsymbol{v} = (-4, 2, 2, 1, 3, 4, -2, -2)$ in the root space of E_8 is E_8-equivalent to others such as $(4, 2, 2, 1, 3, 4, 2, -2)$, $(4, 4, 3, 2, 2, 2, 2, -1)\ldots$ formed by permuting the components of $\boldsymbol{v}$ and by changing the signs of these components in pairs in accordance with (2.3) and (2.4). Still other vectors such as $(-5, 3, 3, 0, 2, 3, -3, -3)$, $(\frac{13}{2}, \frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}), \ldots$ may be shown to be E_8-equivalent to $\boldsymbol{v}$ through the additional application of (2.9). However, the identification of $\boldsymbol{v}_{E_8}$ is by no means trivial.

In order to facilitate this and similar identifications it should be pointed out that the specific selection of coset representatives S_γ made in table 2 for each exceptional Lie group G has been made in such a way that if $\boldsymbol{v}$ is G-dominant then $S_\gamma\boldsymbol{v}$ is H-dominant where H is the appropriate classical Lie subgroup of G. Conversely given that $\boldsymbol{v}$ is H-dominant it is possible to use table 2 to identify a particular coset representative S_γ such that there exists T in the Weyl group W_H of H with the property that $TS_\gamma\boldsymbol{v}$ is G-dominant. The action of the required S_γ on each possible H-dominant vector $\boldsymbol{v}$ for each of the exceptional simple Lie groups G is given in table 3. This may be used, as in the following example, to identify G-dominant vectors. The SO(16)-dominant vector equivalent to $(-4, 2, 2, 1, 3, 4, -2, -2)$ is $\boldsymbol{v}_{SO(16)} = (4, 4, 3, 2, 2, 2, 2, -1)$ for which $v_2 + v_3 + v_4 + v_6 + v_7 - v_8 = 16 > v_1 = 4 \geqslant v_2 + v_3 + v_4 - v_5 - v_6 - v_7 - v_8 = 2$. It follows that the required vector, $S_\gamma\boldsymbol{v}_{SO(16)}$ (from which the corresponding E_8-dominant vector $\boldsymbol{v}_{E_8}$ may be found through the application of some element $T$of $W_{SO(16)}$), is the vector whose components correspond to case (ii) of the entries appropriate to E_8 in table 3, that is $(7, 1, 0, -1, -1, -1, -1, 2)$. Applying permutations and sign changes of pairs of components then gives the required E_8-dominant vector $\boldsymbol{v}_{E_8} = (7, 2, 1, 1, 1, 1, 1, 0)$.

The necessary and sufficient conditions for a vector $\boldsymbol{v}$ to be G-dominant are implicit in table 3. Conditions appropriate to the classical groups have been included for completeness by identifying H with G. The conditions appropriate to each exceptional group G are those specified by case (i) of the entries tabulated, together with the conditions for H-dominance, which are also given.

As explained in I it is useful in developing labels for the irreducible representations of a Lie group G, associated with a semi-simple Lie algebra of rank k, to project each labelling, G-dominant highest weight vector in the k-dimensional root space V of W onto a vector in another k-dimensional subspace Λ of W. This space Λ is spanned by all the vectors $\boldsymbol{e}_i$ with $i = 1, 2, \ldots, d$ except those with $i = m$, where in the notation of I, the values of m are defined so that each vector $\boldsymbol{p}$ of $V^\perp$ takes the form $\boldsymbol{p} = \boldsymbol{e}_i + \boldsymbol{e}_{i+1} + \ldots + \boldsymbol{e}_{i+j-1} + \boldsymbol{e}_m$. The ordering of the basis vectors $\boldsymbol{e}_i$ of W is then such that the projection is onto vectors whose last $(d-k)$-components, with respect to this ordering, are zero. Such a projection, P, may be applied to any vector $\boldsymbol{v}$ of V to give a vector $\boldsymbol{\lambda}$ of

Table 3.

Lie group G	Classical subgroup H	$v \in V$, v H-dominant Classes (i), (ii), ... of vectors v	Components $S_\gamma v$ such that $TS_\gamma v$ is G-dominant for some $T \in W_H$ for each class (i), (ii) ... of H-dominant vectors v		
$SU(k+1)$	$SU(k+1)$	$v_1+v_2+\ldots+v_{k+1}=0$ $v_1 \geqslant v_2 \geqslant \ldots \geqslant v_{k+1}$			
$SO(2k+1)$	$SO(2k+1)$	$v_1 \geqslant v_2 \geqslant \ldots \geqslant v_k \geqslant 0$			
$Sp(2k)$	$Sp(2k)$	$v_1 \geqslant v_2 \geqslant \ldots \geqslant v_k \geqslant 0$			
$SO(2k)$	$SO(2k)$	$v_1 \geqslant v_2 \geqslant \ldots \geqslant v_{k-1} \geqslant v_k \geqslant -v_{k-1}$			
G_2	$SU(3)$	$v_1+v_2+v_3=0$ $v_1 \geqslant v_2 \geqslant v_3$ (i) $v_1 \geqslant 2v_2-v_3$ (ii) $2v_2-v_3>v_1$	(i) v_1 v_2 v_3	(ii) $-v_3$ $-v_2$ $-v_1$	
F_4	$SO(9)$	$v_1 \geqslant v_2 \geqslant v_3 \geqslant v_4 \geqslant 0$ (i) $v_1 \geqslant v_2+v_3+v_4$ (ii) $v_2+v_3+v_4>v_1$	(i) v_1 v_2 v_3 v_4	(ii) $\frac{1}{2}(v_1+v_2+v_3+v_4)$ $\frac{1}{2}(v_1+v_2-v_3-v_4)$ $\frac{1}{2}(v_1-v_2+v_3-v_4)$ $\frac{1}{2}(v_1-v_2-v_3+v_4)$	
E_6	$SU(2)\times SU(6)$	$v_1+v_8=0,\ v_2+v_3+\ldots+v_7=0$ $v_1 \geqslant v_8,\ v_2 \geqslant v_3 \geqslant \ldots \geqslant v_7$ (i) $v_1 \geqslant v_2+v_3+v_4$ (ii) $v_2+v_3+v_4>v_1 \geqslant v_2+v_5+v_6$ (iii) $v_2+v_5+v_6>v_1$	(i) v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8	(ii) $\frac{1}{2}(v_1-v_5-v_6-v_7)$ $\frac{1}{2}(v_1+v_2-v_3-v_4)$ $\frac{1}{2}(v_1-v_2+v_{fi}v_4)$ $\frac{1}{2}(v_1-v_2-v_3+v_4)$ $\frac{1}{2}(-v_1+v_5-v_6-v_7)$ $\frac{1}{2}(-v_1-v_5+v_6-v_7)$ $\frac{1}{2}(-v_1-v_5-v_6+v_7)$ $\frac{1}{2}(-v_1+v_5+v_6+v_7)$	(iii) $\frac{1}{2}(v_2-v_7)$ $v_1+\frac{1}{2}(v_2+v_7)$ $\frac{1}{2}(v_3+v_4+v_5-v_6)$ $\frac{1}{2}(v_3+v_4-v_5+v_6)$ $\frac{1}{2}(v_3-v_4+v_5+v_6)$ $\frac{1}{2}(-v_3+v_4+v_5+v_6)$ $-v_1+\frac{1}{2}(v_2+v_7)$ $\frac{1}{2}(-v_2+v_7)$

E_7	SU(8)	$v_1+v_2+\ldots+v_8=0$	(i) v_1	(ii) $\frac{1}{2}(v_1-v_6-v_7-v_8)$	(iii) $\frac{1}{2}(v_1+v_2+v_3-v_8)$	(iv) $-v_8$
		$v_1\geqslant v_2\geqslant\ldots\geqslant v_8$	v_2	$\frac{1}{2}(v_2-v_3-v_4-v_5)$	$\frac{1}{2}(v_1+v_2-v_3+v_8)$	$-v_7$
		(i) $v_1\geqslant -v_6-v_7-v_8$	v_3	$\frac{1}{2}(-v_2+v_3-v_4-v_5)$	$\frac{1}{2}(v_1-v_2+v_3+v_8)$	$-v_6$
		(ii) $-v_6-v_7-v_8>v_1\geqslant -v_4-v_5-v_8$	v_4	$\frac{1}{2}(-v_2-v_3+v_4-v_5)$	$\frac{1}{2}(v_4+v_5+v_6-v_7)$	$-v_5$
		(iii) $-v_4-v_5-v_8>v_1\geqslant -v_2-v_3-v_8$	v_5	$\frac{1}{2}(-v_2-v_3-v_4+v_5)$	$\frac{1}{2}(v_4+v_5-v_6+v_7)$	$-v_4$
		(iv) $-v_2-v_3-v_8>v_1$	v_6	$\frac{1}{2}(-v_1+v_6-v_7-v_8)$	$\frac{1}{2}(v_4-v_5+v_6+v_7)$	$-v_3$
			v_7	$\frac{1}{2}(-v_1-v_6+v_7-v_8)$	$\frac{1}{2}(-v_4+v_5+v_6+v_7)$	$-v_2$
			v_8	$\frac{1}{2}(-v_1-v_6-v_7+v_8)$	$\frac{1}{2}(-v_1+v_2+v_3+v_8)$	$-v_1$
E_8	SO(16)	$v_1\geqslant v_2\geqslant\ldots\geqslant v_7\geqslant v_8\geqslant -v_7$	(i) v_1	(ii) $\frac{1}{4}(3v_1+v_2+v_3+v_4+v_5+v_6+v_7-v_8)$	(iii) $\frac{1}{2}(v_1+v_2+v_3+v_4)$	
		(i) $v_1\geqslant v_2+v_3+v_4+v_5+v_6+v_7-v_8$	v_2	$\frac{1}{4}(v_1+3v_2-v_3-v_4-v_5-v_6-v_7+v_8)$	$\frac{1}{2}(v_1+v_2-v_3-v_4)$	
		(ii) $v_2+v_3+v_4+v_5+v_6+v_7-v_8>v_1$	v_3	$\frac{1}{4}(v_1-v_2+3v_3-v_4-v_5-v_6-v_7+v_8)$	$\frac{1}{2}(v_1-v_2+v_3-v_4)$	
		$\geqslant v_2+v_3+v_4-v_5-v_6-v_7\pm v_8$	v_4	$\frac{1}{4}(v_1-v_2-v_3+3v_4-v_5-v_6-v_7+v_8)$	$\frac{1}{2}(v_1-v_2-v_3+v_4)$	
		(iii) $v_2+v_3+v_4-v_5-v_6-v_7+v_8>v_1$	v_5	$\frac{1}{4}(v_1-v_2-v_3-v_4+3v_5-v_6-v_7+v_8)$	$\frac{1}{2}(v_5+v_6+v_7+v_8)$	
			v_6	$\frac{1}{4}(v_1-v_2-v_3-v_4-v_5+3v_6-v_7+v_8)$	$\frac{1}{2}(v_5+v_6-v_7-v_8)$	
			v_7	$\frac{1}{4}(v_1-v_2-v_3-v_4-v_5-v_6+3v_7-v_8)$	$\frac{1}{2}(v_5-v_6+v_7-v_8)$	
			v_8	$\frac{1}{4}(-v_1+v_2+v_3+v_4+v_5+v_6+v_7+3v_8)$	$\frac{1}{2}(v_5-v_6-v_7+v_8)$	

Λ in accordance with the prescription:

$$P: \boldsymbol{v} \rightarrow \boldsymbol{\lambda} = \boldsymbol{v} - \sum_{\boldsymbol{p} \in V^{\perp}} v_m \boldsymbol{p} \quad . \tag{3.1}$$

The inverse relationship is:

$$P^{-1}: \boldsymbol{\lambda} \rightarrow \boldsymbol{v} = \boldsymbol{\lambda} - \sum_{\boldsymbol{p} \in V^{\perp}} [(\boldsymbol{\lambda} \cdot \boldsymbol{p})/(\boldsymbol{p} \cdot \boldsymbol{p})]\boldsymbol{p}; \tag{3.2}$$

the consequences of these relations are spelled out in table 4.

It should be stressed that the ordering relations on vectors in W, involving the first non-vanishing difference of components, is applicable directly to vectors $\boldsymbol{v}$ in V but only indirectly to vectors $\boldsymbol{\lambda}$ in Λ. Thus for any two vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in Λ, $\boldsymbol{\lambda} > \boldsymbol{\mu}$ if and only if $P^{-1}\boldsymbol{\lambda} > P^{-1}\boldsymbol{\mu}$. For example in the case of the group SU(3) for which the space $V^{\perp}$ is spanned by $\boldsymbol{p} = \boldsymbol{e}_1 + \boldsymbol{e}_1 + \boldsymbol{e}_3 = (1\ 1\ 1)$ the vectors $\boldsymbol{\lambda} = (1\ 0\ 0)$ and $\boldsymbol{\mu} = (1\ 1\ 0)$ in Λ are such that $(1\ 0\ 0) > (1\ 1\ 0)$ since $P^{-1}\boldsymbol{\lambda} = (\frac{2}{3} -\frac{1}{3} -\frac{1}{3})$ and $P^{-1}\boldsymbol{\mu} = (\frac{1}{3}\ \frac{1}{3} -\frac{2}{3})$, whilst $(\frac{2}{3} -\frac{1}{3} -\frac{1}{3}) > (\frac{1}{3}\ \frac{1}{3} -\frac{2}{3})$ by virtue of the lexicographic ordering in V.

In the same way $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are defined to be G-equivalent if and only if $P^{-1}\boldsymbol{\lambda}$ and $P^{-1}\boldsymbol{\mu}$ are G-equivalent. Moreover $\boldsymbol{\lambda}$ is G-dominant if and only if $P^{-1}\boldsymbol{\lambda}$ is G-dominant. This implies that the G-dominance of vectors $\boldsymbol{\lambda}$ in Λ should be tested through the use of (3.2). Rather than do this from first principles whenever required the necessary and sufficient conditions for a vector $\boldsymbol{\lambda}$ in Λ to be G-dominant have been included in table 4. Finally in this table the further conditions on the components of such a G-dominant vector $\boldsymbol{\lambda}_{\mathrm{G}}$ in Λ for it to label an irreducible representation of G have been displayed. They may be obtained from I. As implied in this previous paper, the highest weight, $\boldsymbol{M}_{\mathrm{G}}$, of the irreducible representation of G labelled by $\boldsymbol{\lambda}_{\mathrm{G}}$, is the G-dominant vector in V defined by

$$\boldsymbol{M}_{\mathrm{G}} = P^{-1}\boldsymbol{\lambda}_{\mathrm{G}}. \tag{3.3}$$

4. Branching rules

The character of the irreducible representation $\boldsymbol{\lambda}_{\mathrm{G}}$ of the semi-simple Lie group G having highest weight $\boldsymbol{M}_{\mathrm{G}}$ is given, in the class specified by parameters $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots \phi_d)$, by the formula due to Weyl (1926):

$$\chi^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{S \in \mathrm{W_G}} \eta_S \exp[\mathrm{i}S(\boldsymbol{R}_{\mathrm{G}} + \boldsymbol{M}_{\mathrm{G}}) \cdot \boldsymbol{\phi}] \Big/ \sum_{S \in \mathrm{W_G}} \eta_S \exp(\mathrm{i}S\boldsymbol{R}_{\mathrm{G}} \cdot \boldsymbol{\phi}) \tag{4.1}$$

where the summations are carried out over all elements S of the Weyl group $\mathrm{W_G}$ and $\boldsymbol{R}_{\mathrm{G}}$ is half the sum of the positive roots of the corresponding Lie algebra g. The G-dominant vectors $\boldsymbol{R}_{\mathrm{G}}$ and $\boldsymbol{M}_{\mathrm{G}}$ lie in the space V which is embedded in W. The vector $\boldsymbol{\phi}$ lies in the same space V, orthogonal where appropriate to $V^{\perp}$.

The branching rules arising from the restriction of group elements of one of the exceptional Lie groups G to its naturally embedded classical Lie subgroup H may be established in many ways. One way involves recognising that the Weyl group $\mathrm{W_H}$ is, for such an embedding, a subgroup of $\mathrm{W_G}$ and that the class parameters of H coincide with those of G. Thus making use of the coset representatives S_γ defined in § 2 it follows that:

$$\chi^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\gamma=1}^{c} \eta_{S_\gamma} \sum_{T \in \mathrm{W_H}} \eta_T \exp[\mathrm{i}TS_\gamma(\boldsymbol{R}_{\mathrm{G}} + \boldsymbol{M}_{\mathrm{G}}) \cdot \boldsymbol{\phi}] \Big/ \sum_{\gamma=1}^{c} \eta_{S_\gamma} \sum_{T \in \mathrm{W_H}} \eta_T \exp(\mathrm{i}TS_\gamma\boldsymbol{R}_{\mathrm{G}} \cdot \boldsymbol{\phi}). \tag{4.2}$$

Table 4.

Lie group G	$\boldsymbol{\lambda} = P\boldsymbol{v}$ for $\boldsymbol{v} \in V$	$\boldsymbol{v} = P^{-1}\boldsymbol{\lambda}$ for $\boldsymbol{\lambda} \in \Lambda$	Conditions for $\boldsymbol{\lambda}$ to be G-dominant	Additional conditions for $\boldsymbol{\lambda}$ to label an irreducible representation
$SU(k+1)$	$\lambda_i = v_i - v_{k+1}$ $i = 1, 2, \ldots, k$	$v_i = \lambda_i - \frac{1}{k+1} l$ $i = 1, 2, \ldots, k$ $v_{k+1} = -\frac{1}{k+1} l$ $l = \lambda_1 + \lambda_2 + \ldots + \lambda_k$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$	λ_i integer for $i = 1, 2, \ldots, k$
$SO(2k+1)$	$\lambda_i = v_i$ $i = 1, 2, \ldots, k$	$v_i = \lambda_i$ $i = 1, 2, \ldots, k$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$	(i) λ_i integer for $i = 1, 2, \ldots, k$ (ii) $\lambda_i - \frac{1}{2}$ integer for $i = 1, 2, \ldots, k$
$Sp(2k)$	$\lambda_i = v_i$ $i = 1, 2, \ldots, k$	$v_i = \lambda_i$ $i = 1, 2, \ldots, k$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0$	λ_i integer for $i = 1, 2, \ldots, k$
$SO(2k)$	$\lambda_i = v_i$ $i = 1, 2, \ldots, k$	$v_i = \lambda_i$ $i = 1, 2, \ldots, k$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{k-1} \geq \lambda_k \geq -\lambda_{k-1}$	(i) λ_i integer for $i = 1, 2, \ldots, k$ (ii) $\lambda_i - \frac{1}{2}$ integer for $i = 1, 2, \ldots, k$
G_2	$\lambda_i = v_i - v_3$ $i = 1, 2$	$v_1 = \frac{1}{3}(2\lambda_1 - \lambda_2)$ $v_2 = \frac{1}{3}(-\lambda_1 + 2\lambda_2)$ $v_3 = \frac{1}{3}(-\lambda_1 - \lambda_2)$	$\lambda_1 \geq \lambda_2 \geq 0$ $\lambda_1 \geq 2\lambda_2$	λ_i integer for $i = 1, 2$
F_4	$\lambda_i = v_i$ $i = 1, 2, 3, 4$	$v_i = \lambda_i$ $i = 1, 2, 3, 4$	$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$ $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4$	(i) λ_i integer for $i = 1, 2, 3, 4$ (ii) $\lambda_i - \frac{1}{2}$ integer for $i = 1, 2, 3, 4$
E_6	$\lambda_1 = v_1 - v_8$ $\lambda_i = v_i - v_7$ $i = 2, 3, \ldots, 6$	$v_1 = -v_8 = \frac{1}{2}\lambda_1$ $v_i = \lambda_i - \frac{1}{6}l$ $i = 2, 3, \ldots, 6$ $v_7 = -\frac{1}{6}l$ $l = \lambda_2 + \lambda_3 + \ldots + \lambda_6$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_6 \geq 0$ $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6$	λ_i integer for $i = 1, 2, \ldots, 6$ $\Sigma_{i=1}^{6} \lambda_i$ even
E_7	$\lambda_i = v_i - v_8$ $i = 1, 2, \ldots, 7$	$v_i = \lambda_i - \frac{1}{8}l$ $i = 1, 2, \ldots, 7$ $v_8 = -\frac{1}{8}l$ $l = \lambda_1 + \lambda_2 + \ldots + \lambda_7$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_7 \geq 0$ $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7$	λ_i integer for $i = 1, 2, \ldots, 7$ $\Sigma_{i=1}^{7} \lambda_i$ even
E_8	$\lambda_i = v_i$ $i = 1, 2, \ldots, 8$	$v_i = \lambda_i$ $i = 1, 2, \ldots, 8$	$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_7 \geq \lambda_8 \geq -\lambda_7$ $\lambda_1 \geq \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 - \lambda_8$	(i) λ_i integer for $i = 1, 2, \ldots, 8$ (ii) $\lambda_i - \frac{1}{2}$ integer for $i = 1, 2, \ldots, 8$ $\Sigma_{i=1}^{8} \lambda_i$ even

Dividing the numerator and denominator by

$$\sum_{T\in W_H} \eta_T \exp(\mathrm{i}T\boldsymbol{R}_H \cdot \boldsymbol{\phi})$$

allows this character (4.2) to be written as a quotient of sums of characters of H. To be precise:

$$\chi^{\boldsymbol{\lambda}_G}(\boldsymbol{\phi}) = \sum_{\gamma=1}^{c} \eta_{S_\gamma} \chi^{\boldsymbol{\sigma}_H^{(\gamma)}}(\boldsymbol{\phi}) \Big/ \sum_{\gamma=1}^{c} \eta_{S_\gamma} \chi^{\boldsymbol{\tau}_H^{(\gamma)}}(\boldsymbol{\phi}), \tag{4.3}$$

where $\boldsymbol{\sigma}_H^{(\gamma)}$ and $\boldsymbol{\tau}_H^{(\gamma)}$ label the irreducible representations of H having highest weights $S_\gamma(\boldsymbol{R}_G+\boldsymbol{M}_G)-\boldsymbol{R}_H$ and $S_\gamma\boldsymbol{R}_G-\boldsymbol{R}_H$ respectively, i.e.

$$\boldsymbol{\sigma}_H^{(\gamma)} = P(S_\gamma(\boldsymbol{R}_G+\boldsymbol{M}_G)-\boldsymbol{R}_H) \tag{4.4}$$

and

$$\boldsymbol{\tau}_H^{(\gamma)} = P(S_\gamma\boldsymbol{R}_G-\boldsymbol{R}_H). \tag{4.5}$$

It should be stressed that the particular choice made of coset representatives, S_γ, whose action is defined in table 3 is now highly advantageous since this action on the G-dominant vectors $\boldsymbol{R}_G+\boldsymbol{M}_G$ and $\boldsymbol{M}_G$ is guaranteed to give H-dominant vectors. Moreover the vectors $S_\gamma(\boldsymbol{R}_G+\boldsymbol{M}_G)-\boldsymbol{R}_H$ and $S_\gamma\boldsymbol{R}_G-\boldsymbol{R}_H$ are also H-dominant as required for $\boldsymbol{\sigma}_H^{(\gamma)}$ and $\boldsymbol{\tau}_H^{(\gamma)}$, defined by (4.4) and (4.5), to be irreducible representation labels as they stand, without modification.

Thus making use of $\boldsymbol{R}_G$ and $\boldsymbol{R}_H$ as given in table 1, the action of S_γ as defined in table 2, and the projection P defined in table 4 it is possible to write out the expansion (4.3) explicitly for each exceptional Lie group G.

This has already been done (King and Al-Qubanchi 1978) for G_2, and yields in the present notation:

$$\chi^{(\lambda_1\lambda_2)}(\boldsymbol{\phi}) = (\chi^{\{\lambda_1+1,\lambda_2\}}(\boldsymbol{\phi}) - \chi^{\{\lambda_1+1,\lambda_1-\lambda_2+1\}}(\boldsymbol{\phi}))/(\chi^{\{1\}}(\boldsymbol{\phi})-\chi^{\{1^2\}}(\boldsymbol{\phi})) \tag{4.6}$$

where the brackets ( ) and { } serve to distinguish between the characters of G_2 and its subgroup SU(3).

Similarly in the case of F_4 the same formula (4.3) gives

$$\begin{aligned}\chi^{(\lambda_1\lambda_2\lambda_3\lambda_4)}(\boldsymbol{\phi}) = \{&\chi^{[\lambda_1\lambda_2\lambda_3\lambda_4]}(\boldsymbol{\phi}) \\ &-\chi^{[\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4)+\frac{3}{2},\frac{1}{2}(\lambda_1+\lambda_2-\lambda_3-\lambda_4)+\frac{1}{2},\frac{1}{2}(\lambda_1-\lambda_2+\lambda_3-\lambda_4)+\frac{1}{2},\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3+\lambda_4)+\frac{1}{2}]}(\boldsymbol{\phi}) \\ &+\chi^{[\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3-\lambda_4)+1,\frac{1}{2}(\lambda_1+\lambda_2-\lambda_3+\lambda_4)+1,\frac{1}{2}(\lambda_1-\lambda_2+\lambda_3+\lambda_4)+1,\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4)]}(\boldsymbol{\phi})\} \\ &\times 1/\{\chi^{[2]}(\boldsymbol{\phi}) - \chi^{[\frac{3}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}]}(\boldsymbol{\phi}) + \chi^{[1^3]}(\boldsymbol{\phi})\}\end{aligned} \tag{4.7}$$

where now the brackets ( ) and [ ] serve to distinguish between the characters of F_4 and its subgroup SO(9).

Formulae similar to (4.6) and (4.7) may be written down for E_6, E_7 and E_8 but they contain in the numerator and denominator c terms where $c=36$, 72 and 135 respectively. Nonetheless they do serve to define exceptional group characters in terms of classical group characters, and in principle they may be used to determine the corresponding branching multiplicities $B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G}$, appropriate to the restriction from G to H, which are defined by the expansion:

$$\chi^{\boldsymbol{\lambda}_G}(\boldsymbol{\phi}) = \sum_{\boldsymbol{\mu}_H} B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G} \chi^{\boldsymbol{\mu}_H}(\boldsymbol{\phi}). \tag{4.8}$$

By means of a stroke of good fortune in the case of G_2, which does not appear to apply to the remaining exceptional Lie groups, the quotient (4.6) may be evaluated explicitly (King and Al-Qubanchi 1978) in the form (4.8) to yield the branching multiplicities for the restriction from G_2 to SU(3).

In general these multiplicities may be found by systematically building up the numerator of (4.3), working from the highest representation $\boldsymbol{\sigma}_{\mathrm{H}}^{(\gamma)}$ downwards, by multiplying the denominator of (4.3) by characters $\chi^{\mu_{\mathrm{H}}}(\boldsymbol{\phi})$ chosen so as to give the required terms, after appropriate cancellations have been made involving the factors η_{S_γ}. This procedure involves only a knowledge of the Kronecker product multiplicities of the classical group H defined by:

$$\chi^{\tau_{\mathrm{H}}}(\boldsymbol{\phi})\chi^{\sigma_{\mathrm{H}}}(\boldsymbol{\phi}) = \sum_{\rho_{\mathrm{H}}} K_{\rho_{\mathrm{H}}}^{\tau_{\mathrm{H}}\sigma_{\mathrm{H}}}\chi^{\rho_{\mathrm{H}}}(\boldsymbol{\phi}). \tag{4.9}$$

Some results for the branching multiplicities for the restriction from F_4 to SO(9) have been obtained in this way from (4.7) and given elsewhere (Al-Qubanchi 1978). However the method is very tedious, comparing most unfavourably with that used by Wybourne and Bowick (1977) based on the method of elementary multiplets due to Sharp and Lam (1969). An alternative approach to this branching rule problem arises, as shown in the next section, as a by-product of a method of calculating weight multiplicities.

5. Weight multiplicities

There is another expansion of the character (4.1) of an irreducible representation $\boldsymbol{\lambda}_{\mathrm{G}}$ which serves to define the weights $\boldsymbol{m}$ of this representation. This expansion takes the form:

$$\chi^{\lambda_{\mathrm{G}}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}} \exp(\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\phi}), \tag{5.1}$$

where the coefficient $M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}}$ is the multiplicity of the weight $\boldsymbol{m}$ in the representation $\boldsymbol{\lambda}_{\mathrm{G}}$.

This expansion may be effected in many ways. For the classical Lie groups an extensive tabulation of weight multiplicities already exists (King and Plunkett 1976). For the exceptional Lie groups G it is convenient to make use of the branching rule, (4.8), associated with the restriction from G to a classical Lie subgroup H, together with the character formula (5.1) appropriate to H:

$$\chi^{\mu_{\mathrm{H}}}(\boldsymbol{\phi}) = \sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\mu_{\mathrm{H}}} \exp(\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\phi}). \tag{5.2}$$

Comparison with (5.1) then yields:

$$M_{\boldsymbol{m}}^{\lambda_{\mathrm{G}}} = \sum_{\mu_{\mathrm{H}}} B_{\mu_{\mathrm{H}}}^{\lambda_{\mathrm{G}}} M_{\boldsymbol{m}}^{\mu_{\mathrm{H}}}. \tag{5.3}$$

If the branching multiplicities are known as well as the weight multiplicities of the classical group H this provides a very simple way of determining the weights and the weight multiplicities of the exceptional group G. It should be noted that (5.3) implies that every weight of a representation of G is necessarily the weight of some representation of H. This is a consequence of the selection of H as a subgroup embedded naturally in G which allows the class parameters for G and H in (5.1) and (5.2) to be identified. It is now possible to exploit the fact that the Weyl group W_H is a subgroup of W_G.

The dependence of the character formula (4.1) on the elements S of the Weyl group W_G is such that, on expanding in the form (5.1), it is clear that

$$M^{\lambda_G}_{Sm} = M^{\lambda_G}_{m} \tag{5.4}$$

for all S in W_G. In the case for which S is a coset representative, S_γ, of W_G with respect to W_H this yields, through (5.3), the non-trivial constraints:

$$M^{\lambda_G}_{m} = M^{\lambda_G}_{S_\gamma m} = \sum_{\mu_H} B^{\lambda_G}_{\mu_H} M^{\mu_H}_{S_\gamma m} \tag{5.5}$$

for $\gamma = 1, 2, \ldots, c$, where $M^{\mu_H}_{S_\gamma m}$ has *a priori* no connection with $M^{\mu_H}_{m}$.

Given any particular weight $\boldsymbol{m}$ it is only necessary by virtue of (5.4) to evaluate the multiplicity of the corresponding G-dominant weight $\boldsymbol{m}_G$. Then (5.5) gives:

$$M^{\lambda_G}_{m_G} = \sum_{\mu_H} B^{\lambda_G}_{\mu_H} M^{\mu_H}_{S_\gamma m_G} \tag{5.6}$$

for all coset representatives S_γ, which, as pointed out in § 3, have been chosen so as to ensure that $S_\gamma \boldsymbol{m}_G$ is H-dominant. This relationship between dominant weight multiplicities of G and those of H, involving the branching multiplicities, serves as a powerful weapon in tackling the problems of evaluating both weight multiplicities of G and the branching multiplicities.

Since the weights $\boldsymbol{m}_G$ and $S_\gamma \boldsymbol{m}_G$ appearing in (5.6) are G and H-dominant respectively they are necessarily highest weights of some irreducible representations of G and H: in fact those labelled by $\boldsymbol{\nu}_G = P\boldsymbol{m}_G$ and $\boldsymbol{\sigma}^\gamma_H = PS_\gamma \boldsymbol{m}_G$, where in the labelling scheme adopted here, the projection operator P is the same for both the groups G and H. It is convenient, without it is hoped any great risk of confusion, to define:

$$M^{\lambda_G}_{\nu_G} = M^{\lambda}_{m_G} \qquad \text{and} \qquad M^{\mu_H}_{\sigma_{\gamma H}} = M^{\mu_H}_{S_\gamma m_G} \tag{5.7}$$

so that

$$M^{\lambda_G}_{\nu_G} = \sum_{\mu_H} B^{\lambda_G}_{\mu_H} M^{\mu_H}_{\sigma_{\gamma H}} \qquad \text{for } \gamma = 1, 2, \ldots, c \tag{5.8}$$

where

$$\boldsymbol{\sigma}^\gamma_H = PS_\gamma P^{-1} \boldsymbol{\nu}_G. \tag{5.9}$$

Thus in order to maximise the benefit obtained from (5.8) it is useful to tabulate for each representation label $\boldsymbol{\nu}_G$ the complete list of representation labels $\boldsymbol{\sigma}^\gamma_H$ defined by (5.9). Unfortunately P and S_γ do not in general commute so it is necessary to make use of both table 2, defining the action of S_γ, and table 4 defining that of P, in constructing such a list. Some results are displayed in table 5.

Only for G_2 is it trivial to derive the general result. In this case:

$$PS_1P^{-1}(\nu_1\nu_2) = PP^{-1}(\nu_1\nu_2) = (\nu_1\nu_2) \tag{5.10}$$

and

$$\begin{aligned} PS_2P^{-1}(\nu_1\nu_2) &= PS_{(e_1-2e_2+e_3)/3}(\tfrac{2}{3}\nu_1 - \tfrac{1}{3}\nu_2,\ -\tfrac{1}{3}\nu_1 + \tfrac{2}{3}\nu_2,\ -\tfrac{1}{3}\nu_1 - \tfrac{1}{3}\nu_2) \\ &= P(\tfrac{1}{3}\nu_1 + \tfrac{1}{3}\nu_2,\ \tfrac{1}{3}\nu_1 - \tfrac{2}{3}\nu_2,\ -\tfrac{2}{3}\nu_1 + \tfrac{1}{3}\nu_2) \\ &= (\nu_1,\ \nu_1 - \nu_2) \end{aligned} \tag{5.11}$$

where use has been made of (3.1) and (3.2) with $p = (1, 1, 1)$, and of (2.8). This merely corresponds to the fact that the highest weights of the mutually contragredient representations $\{\nu_1, \nu_2\}$ and $\{\nu_1, \nu_1 - \nu_2\}$ of SU(3) are G_2-equivalent. It is obviously true

Table 5.

G-dominant vector ν_G in Λ	Set of H-dominant vectors $PSP^{-1}\,\nu_G$ for $S \in W_G$ which are G-equivalent to ν_G						
$G = G_2$		$H = SU(3)$					
ν_1, ν_2	ν_1, ν_2	$\nu_1, \nu_1 - \nu_2$					
$G = F_4$		$H = SO(9)$					
0	0						
1	1	Δ					
1^2	1^2						
$\Delta; 1$	$\Delta; 1$	1^3					
2	2	1^4					
21	21	$\Delta; 1^2$					
21^2	21^2						
2^2	2^2						
$\Delta; 2$	$\Delta; 2$	21^3	$\Delta; 1^3$				
$\Delta; 21$	$\Delta; 21$	2^21					
3	3	$\Delta; 1^4$					
31	31	2^21^2					
31^2	31^2	$\Delta; 21^2$					
31^3	31^3	2^3					
32	32	$\Delta; 2^2$					
321	321						
3^2	3^2						
$G = E_6$		$H = SU(2) \otimes SU(6)$					
$0:0$	$0:0$						
$1:1$	$1:1$	$0:1^4$					
$1:1^5$	$1:1^5$	$0:1^2$					
$2:0$	$2:0$	$1:1^3$	$0:21^4$				
$2:1^2$	$2:1^2$	$1:21^3$	$0:2$	$0:2^31^2$			
$2:1^4$	$2:1^4$	$1:2^21^3$	$0:21^2$	$0:2^5$			
$2:2$	$2:2$	$0:2^4$					
$2:21^4$	$2:21^4$	$1:21$	$1:2^41$	$0:2^21^2$			
$2:2^5$	$2:2^5$	$0:2^2$					
$3:1$	$3:1$	$2:21^2$	$1:31^4$	$1:2^31$	$0:32^31$		
$3:1^3$	$3:1^3$	$2:2^21^2$	$1:32^21^2$	$0:31^3$	$0:2^3$	$0:3^22^3$	
$3:1^5$	$3:1^5$	$2:2^31^2$	$1:2^21$	$1:32^4$	$0:321^3$		
$3:21$	$3:21$	$2:31^3$	$1:3$	$1:32^3$	$0:3^321$		
$3:21^3$	$3:21^3$	$2:321^3$	$2:2^4$	$1:31^2$	$1:3^22^21$	$0:32^21$	$0:3^42$
$3:2^21^3$	$3:2^21^3$	$2:2^2$	$2:32^31$	$1:321^2$	$1:3^32^2$	$0:31$	$0:3^221^2$
$3:2^41$	$3:2^41$	$2:32^3$	$1:31^3$	$1:3^5$	$0:321$		
$3:3$	$3:3$	$0:3^4$					
$3:31^4$	$3:31^4$	$2:31$	$1:3^41$	$0:3^22^2$			
$3:32^4$	$3:32^4$	$2:3^42$	$1:32$	$0:3^21^2$			
$3:3^5$	$3:3^5$	$0:3^2$					

Table 5—continued

G-dominant vector ν_G in Λ	Set of H-dominant vectors $PSP^{-1}\nu_G$ for $S \in W_G$ which are G-equivalent to ν_G					
$G = E_7$		$H = SU(8)$				
0	0					
1^2	1^2	1^6				
2	2	2^7				
21^2	21^2	2^21^4	2^51^2			
21^6	21^6	1^4				
2^2	2^2	2^6				
31	31	321^3	3^221^4	32^41	2^32^31	3^62
31^3	31^3	32^21^3	2^3	3^22^31	2^5	3^42^3
31^5	31^5	32^31^3	2^31^2	3^22^5		
321	321	3^21^4	3^22^4	3^521		
321^5	321^5	32^51	21^4	2^21^2	2^41^2	
3^2	3^2	3^6				
42^6	42^6	2^4				
$G = E_8$		$H = SO(16)$				
0	0					
1^2	1^2	$(\boldsymbol{\Delta})_+$				
2	2	$(\boldsymbol{\Delta}; 1)_-$	1^4			
21^2	21^2	$(\boldsymbol{\Delta}; 1^2)_+$	1^6			
2^2	2^2	$(1^8)_+$				
$(\boldsymbol{\Delta}; 2)_+$	$(\boldsymbol{\Delta}; 2)_+$	21^4	$(\boldsymbol{\Delta}; 1^3)_-$	$(1^8)_-$		
31	31	21^6	2^21^2	$(\boldsymbol{\Delta}; 21)_-$	$(\boldsymbol{\Delta}; 1^4)_+$	
31^3	31^3	$(\boldsymbol{\Delta}; 21^2)_+$	2^3	2^21^4	$(\boldsymbol{\Delta}; 1^5)_-$	
321	321	$(\boldsymbol{\Delta}; 2^2)_+$	$(2^21^6)_+$	$(\boldsymbol{\Delta}; 1^6)_+$		
3^2	3^2	$(\boldsymbol{\Delta}; 1^8)_+$				

in this case that the weight multiplicities of G_2 derived earlier (King and Al-Qubanchi 1978) using (5.8) with $c = 1$ are entirely consistent with (5.8) in the case $c = 2$ since the branching rule from G_2 to SU(3) involves mutually contragredient pairs of representations along with self-contragredient representations of SU(3).

However, in other cases the constraints (5.8) with $c \neq 1$ are much more important. To see this it is instructive to consider the problem in the case when $G = E_8$ and $H = SO(16)$. The defining representation of E_8 is the adjoint representation (1^2). The weights of this representation are precisely the roots themselves, each with multiplicity 1, together with the null vector whose multiplicity is the rank, 8, of E_8. All the roots of E_8 are necessarily E_8-equivalent, but as indicated in table 5 they include two SO(16)-dominant weights (1^2) and $(\boldsymbol{\Delta})_+$ which are not SO(16)-equivalent. This implies, after a dimensionality check to ensure that all weights have been included, that on restriction from E_8 to SO(16):

$$E_8 \to SO(16) \qquad (1^2) \to [1^2] + [\boldsymbol{\Delta}]_+.$$

The relevant portion of the weight multiplicity table (King and Plunkett 1976) of

SO(16) takes the form:

	(0)	(1^2)	$(\Delta)_+$
$[1^2]$	8	1	
$[\Delta]_+$			1
	8	1	1

This confirms the weight multiplicities of E_8:

$$M^{(1^2)}_{(0)}=8 \qquad M^{(1^2)}_{(1^2)}=M^{(1^2)}_{(\Delta)_+}=1.$$

Similarly in the case of the representation (2) of E_8 the SO(16)-dominant weights (2), $(\Delta;1)_-$ and (1^4) are E_8-equivalent as indicated in table 5. A dimension check then verifies the validity of the branching rule:

$$E_8\to SO(16) \qquad (2)\to[2]+[\Delta;1]_-+[1^4].$$

Correspondingly the weight multiplicity table of SO(16) furnishes the results:

	(0)	(2)	(1^2)	(1^4)	$(\Delta;0)_+$	$(\Delta;1)_-$
[2]	7	1	1			
$[\Delta;1]_-$					7	1
$[1^4]$	28		6	1		
	35	1	7	1	7	1

so that

$$M^{(2)}_{(0)}=35$$
$$M^{(2)}_{(1^2)}=M^{(2)}_{(\Delta;0)_+}=7$$
$$M^{(2)}_{(2)}=M^{(2)}_{(\Delta;1)_-}=M^{(2)}_{(1^4)}=1$$

where the equalities in weight multiplicities of E_8 are consistent with the requirements of E_8-equivalence.

Continuing, the representation (21^2) of E_8 has as its highest weight (21^2) which is E_8-equivalent to the SO(16)-dominant weights $(\Delta;1^2)$ and (1^6), so that on restriction:

$$E_8\to SO(16) \qquad (21^2)\to[21^2]+[\Delta;1^2]_++[1^6]+\dots.$$

The corresponding SO(16) weight multiplicities are:

	(0)	(2)	(1^2)	(21^2)	(1^4)	(1^6)	$(\Delta;0)_+$	$(\Delta;1)_-$	$(\Delta;1^2)_+$
$[21^2]$	76	7	19	1	3				
$[\Delta;1^2]_+$							28	6	1
$[1^6]$	56		15		4	1			
	132	7	34	1	7	1	28	6	1

Table 6. F_4 dominant weight multiplicities M_{ν}^{λ}.

	c_ν	1	24	24	96	24	144	96	24	192	288	24	144	288	96	144	192	24
d_λ	$\lambda \backslash \nu$	0	1	1^2	$\Delta;1$	2	21	21^2	2^2	$\Delta;2$	$\Delta;21$	3	31	31^2	31^3	32	321	3^2
1	0	1																
26	1	2	1															
52	1^2	4	1	1														
273	$\Delta;1$	9	5	2	1													
324	2	12	5	3	1	1												
1053	21	21	14	6	4	1	1											
1274	21^2	26	13	10	4	3	1	1										
1053	2^2	21	9	8	3	3	1	1	1									
4096	$\Delta;2$	64	40	24	14	8	4	2	0	1								
8424	$\Delta;21$	96	66	40	27	15	10	5	2	3	1							
2652	3	36	24	13	9	5	3	1	0	1	0	1						
10829	31	125	77	56	32	23	12	9	3	4	1	1	1					
19278	31^2	174	126	80	57	33	24	13	4	9	3	3	1	1				
19448	31^3	176	117	84	54	39	23	16	6	9	3	3	2	1	1			
17901	32	141	106	66	50	29	23	12	6	9	4	3	1	1	0	1		
29172	321	228	147	118	72	58	34	28	14	14	6	5	5	2	2	1	1	
12376	3^2	88	53	47	27	24	14	13	8	6	3	3	3	1	1	1	1	1

Table 7. E_6 dominant weight multiplicities M^{λ}_{ν}.

	c_ν	1	72	270	720	432	432	27	27
d_λ	$\lambda \backslash \nu$	0:0	2:0	$2:21^4$	$3:1^3$	3:21	$3:2^41$	3:3	$3:3^5$
1	0:0	1							
78	2:0	6	1						
650	$2:21^4$	20	5	1					
2925	$3:1^3$	45	15	4	1				
5824	3:21	64	24	8	2	1			
5824	$3:2^41$	64	24	8	2	0	1		
3003	3:3	24	10	4	1	1	0	1	
3003	$3:3^5$	24	10	4	1	0	1	0	1

	c_ν	27	216	27	432	1080	270
d_λ	$\lambda \backslash \nu$	1:1	$2:1^4$	$2:2^5$	3:1	$3:2^21^3$	$3:31^4$
27	1:1	1					
351	$2:1^4$	5	1				
351	$2:2^5$	4	1	1			
1728	3:1	16	4	0	1		
7371	$3:2^21^3$	44	15	5	4	1	
7722	$3:31^4$	40	14	4	5	1	1

	c_ν	27	216	27	432	1080	270
d_λ	$\lambda \backslash \nu$	$1:1^5$	$2:1^2$	2:2	$3:1^5$	$3:21^3$	$3:32^4$
27	$1:1^5$	1					
351	$2:1^2$	5	1				
351	2:2	4	1	1			
1728	$3:1^5$	16	4	0	1		
7371	$3:21^3$	44	15	5	4	1	
7722	$3:32^4$	40	14	4	5	1	1

Clearly:

$$M^{(21^2)}_{(21^2)} = M^{(21^2)}_{(\Delta;1^2)_+} = M^{(21^2)}_{(1^6)} = 1$$

as required. However, from (5.4)

$$M^{(21^2)}_{(2)} = M^{(21^2)}_{(\Delta;1)_-} = M^{(21^2)}_{(1^4)}.$$

In order for this relation to be satisfied it is necessary for the representation $[\Delta; 1]_-$ of SO(16) to appear in the restriction. This gives the additional weights and running totals:

	(0)	(2)	(1^2)	(21^2)	(1^4)	(1^6)	$(\Delta;0)_+$	$(\Delta;1)_-$	$(\Delta;1^2)_+$
$[\Delta;1]_-$							7	1	
	132	7	34	1	7	1	35	7	1

so that

$$M^{(21^2)}_{(2)} = M^{(21^2)}_{(\Delta;\,1)_-} = M^{(21^2)}_{(1^4)} = 7.$$

Now, however, it is required that

$$M^{(21^2)}_{(1^2)} = M^{(21^2)}_{(\Delta;\,0)_+}$$

which may be satisfied through the inclusion of $[1^2]$ to yield:

	(0)	(2)	(1^2)	(21^2)	(1^4)	(1^6)	$(\Delta;0)_+$	$(\Delta;1)_-$	$(\Delta;1^2)_+$
$[1^2]$	8		1						
	140	7	35	1	7	1	35	7	1

so that

$$M^{(21^2)}_{(1^2)} = M^{(21^2)}_{(\Delta;0)_+} = 7.$$

A dimensional check then confirms that the branching rule

$$E_8 \rightarrow SO(16) \qquad (21^2) \rightarrow [21^2] + [\Delta;\, 1^2]_+ + [\Delta;\, 1]_- + [1^6] + [1^2]$$

is now complete. Hence

$$M^{(21^2)}_{(0)} = 140$$

completes the evaluation of weight multiplicities of the representation (21^2) of E_8.

Table 8. E_7 dominant weight multiplicities M^{λ}_{ν}.

d_λ	c_ν	1	126	756	56	2016	126	4032
	$\lambda \backslash \nu$	0	21^6	21^2	2^2	31^5	42^6	31
1	0	1						
133	21^6	7	1					
1539	21^2	27	6	1				
1463	2^2	21	5	1	1			
8645	31^5	77	22	5	0	1		
7371	42^6	63	17	4	0	1	1	
40755	31	225	69	23	6	5	0	1

d_λ	c_ν	56	576	1512	4032	1512	56
	$\lambda \backslash \nu$	1^2	2	321^5	31^3	321	3^2
56	1^2	1					
912	2	6	1				
6480	321^5	27	6	1			
27664	31^3	71	21	5	1		
51072	321	111	21	10	4	1	
24320	3^2	45	15	5	1	1	1

Table 9. E_8 dominant weight multiplicities M_ν^λ.

	c_ν	1	240	2160	6720	240	17280	30240	60480	13440	240
d_λ	λ \ ν	0	1^2	2	21^2	2^2	$(\Delta; 2)_+$	31	31^3	321	3^2
1	0	1									
248	1^2	8	1								
3875	2	35	7	1							
30380	21^2	140	35	7	1						
27000	2^2	120	29	6	1	1					
147250	$(\Delta; 2)_+$	370	111	29	6	0	1				
779247	31	1407	455	133	34	7	7	1			
2450240	31^3	2960	1056	350	105	27	28	6	1		
4096000	321	4480	1624	552	174	56	48	12	2	1	
1763125	3^2	1765	645	224	74	29	21	6	1	1	1

Table 10. Branching multiplicities B^{λ}_{μ} for $E_8 \to SO(16)$.

	E_8 d_λ	1	248	3875	30380	27000	147250	779247	2450240	4096000	1763125
SO(16) d_μ	μ \ λ	0	1^2	2	21^2	2^2	$(\boldsymbol{\Delta};2)_+$	31	31^3	321	3^2
1	0	1				1					
120	1^2		1		1			1		1	1
128	$(\boldsymbol{\Delta})_+$		1			1		1		1	1
135	2			1			1		1		
1920	$(\boldsymbol{\Delta};1)_-$			1	1		1	1	1	1	
1820	1^4			1		1		1	1	1	
7020	21^2				1		1	1	1	1	
13312	$(\boldsymbol{\Delta};1^2)_+$				1	1		1	1	2	1
8008	1^6				1			1		1	1
5304	2^2					1			1	1	
6435	$(1^8)_+$					1				1	1
15360	$(\boldsymbol{\Delta};2)_+$						1	1	1		
60060	21^4						1	1	1	1	
56320	$(\boldsymbol{\Delta};1^3)_-$						1	1	1	1	
6435	$(1^8)_-$						1		1		
8925	31							1			
162162	21^6							1	1	1	
141372	2^21^2							1		1	1
141440	$(\boldsymbol{\Delta};21)_-$							1	1	1	
161280	$(\boldsymbol{\Delta};1^4)_+$							1		1	1
176800	31^3								1		
670208	$(\boldsymbol{\Delta};21^2)_+$								1	1	
89760	2^3								1		
716040	2^21^4								1	1	
326144	$(\boldsymbol{\Delta};1^5)_-$								1		
344064	321									1	
595595	$(2^21^6)_+$									1	1
524160	$(\boldsymbol{\Delta};2^2)_+$									1	1
465920	$(\boldsymbol{\Delta};1^6)_+$									1	
129675	3^2										1
183040	$(\boldsymbol{\Delta};1^8)_+$										1

In this way branching multiplicity and weight multiplicity tables may be built up simultaneously with one being used to check the other. Both the branching multiplicities for $G_2 \to SU(3)$ and the weight multiplicities of G_2 have been given earlier (King and Al-Qubanchi 1978) whilst the branching multiplicities for $F_4 \to SO(9)$, $E_6 \to SU(2) \otimes SU(6)$, and $E_7 \to SU(8)$ have been extensively tabulated by Wybourne and Bowick (1977). The weight multiplicity table of F_4 due to Veldkamp (1970) is extended to give the results of table 6, whilst the weight multiplicities of E_6, E_7 and E_8 are given in tables 7, 8 and 9. It should be pointed out that the last table is not as extensive as the remarkable table produced by Freudenthal (1954b) for E_8 as an illustration of the power of his recurrence relation (Freudenthal 1954a) for calculating weight multiplicities. However, table 9 serves to present the results in the notation described in I, whilst table 10 considerably extends the list of branching multiplicities for $E_8 \to SO(16)$ given by Freudenthal (1956).

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